Chapter 8

Analysis of variance

Exercises

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8 Analysis of variance

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8.1 Environment action plans

|||| Exercise 8.1 Environment action plans

To assess whether the two Danish aquatic environment action plans affected nitrogen levels in a river, nitrogen concentration (measured in $\mathrm{g/m^3}$) was recorded at three time points:

- 1998: immediately before the first action plan,
- 2003: immediately before the second action plan,
- 2011: after both action plans had been implemented.

At each time point, the nitrogen concentration was measured six times. The observed concentrations are shown below:

	N_{1998}	N_{2003}	N_{2011}
	5.01	5.59	3.02
	6.23	5.13	4.76
	5.98	5.33	3.46
	5.31	4.65	4.12
	5.13	5.52	4.51
	5.65	4.92	4.42
Mean	5.552	5.190	4.048

The total sum of squares is given by: $SST = 11.4944$.

A one-way ANOVA was performed to compare the mean nitrogen concentrations across the three years. The resulting ANOVA table is shown below, where some values have been substituted with letters:

```
fit = smf.ols('nitrogen ~ year', data=D).fit()
print(sm.stats.anova_lm(fit))

            df   sums_q  mean_sq          F    PR(>F)
Year         A        B        C          U         V
Residuals    D   4.1060        E        NaN       NaN
```

a) Determine the values represented by the letters A–D.

b) At $\alpha = 0.05$, what critical value is used for the hypothesis test (represented by the letters U and V)?

c) Using a significance level of $\alpha = 0.05$, test whether the mean nitrogen concentration differs between years. State the null hypothesis, calculate the test statistic and p-value, and draw an appropriate conclusion.

d) Compute the 90% confidence interval for the mean difference between 2011 and 1998.

Use Python to compute quantiles and probabilities from relevant probability distribution(s).

Exam-style questions

e) A different one-way ANOVA with $k = 4$ groups and $n = 24$ total observations gives $SST = 85.6$ and $SSE = 52.3$. What are $SS(Tr)$, $MS(Tr)$, and MSE?

1) $SS(Tr) = 137.9, MS(Tr) = 45.97, MSE = 2.615$
2) $SS(Tr) = 33.3, MS(Tr) = 11.1, MSE = 2.615$
3) $SS(Tr) = 33.3, MS(Tr) = 8.325, MSE = 2.615$
4) $SS(Tr) = 33.3, MS(Tr) = 11.1, MSE = 2.18$
5) $SS(Tr) = 33.3, MS(Tr) = 11.1, MSE = 17.43$

8.2 Environment action plans (part 2)

|||| Exercise 8.2 Environment action plans (part 2)

This exercise uses the same nitrogen concentration data as Exercise 8.1 (measurements in g/m^3 at three time points: 1998, 2003, 2011; 6 replicates each).

The group variances are added to the table:

	N_{1998}	N_{2003}	N_{2011}
	5.01	5.59	3.02
	6.23	5.13	4.76
	5.98	5.33	3.46
	5.31	4.65	4.12
	5.13	5.52	4.51
	5.65	4.92	4.42
Mean	5.5517	5.1900	4.0483
Variance	0.2366	0.1313	0.4533

Further, the overall mean is: $\bar{y} = 4.93$.

a) Compute the three sums of squares (SST, $SS(Tr)$, and SSE) using the group means, group variances, and overall mean. Show the formulas explicitly.

The raw data can be read into Python by the following code:

```
nitrogen = np.array([
5.01, 5.59, 3.02,
6.23, 5.13, 4.76,
5.98, 5.33, 3.46,
5.31, 4.65, 4.12,
5.13, 5.52, 4.51,
5.65, 4.92, 4.42
])
year = pd.Categorical(np.tile(["1998", "2003", "2011"], 6))
D = pd.DataFrame({"nitrogen": nitrogen, "year": year})
```

b) Find the SST value in Python using the sample variance function ("`np.var(ddof=1)`").

c) Run the ANOVA in Python and produce the ANOVA table.

d) Do a complete post hoc analysis, comparing all 3 years pairwise.

e) Use Python to do model validation by residual analysis.

8.3 Plastic film

|||| Exercise 8.3 Plastic film

A company is starting a production of a new type of patch. For the product a thin plastic film is to beused. Samples of products were received from 5 possible suppliers. Each sample consisted of 20 measurements of the film thickness and the following data were found:

	Average film thickness ($\bar{x}$ in μm)	Sample standard deviation (s in μm)
Supplier 1	31.4	1.9
Supplier 2	30.6	1.6
Supplier 3	30.5	2.2
Supplier 4	31.3	1.8
Supplier 5	29.2	2.2

From the usual calculations for a one-way analysis of variance the following is obtained:

Source	Degrees of freedom	Sum of Squares
Supplier	4	$SS(Tr) = 62$
Error	95	$SSE = 362.71$

a) Is there a significant ($\alpha = 5\%$) difference between the mean film thicknesses for the suppliers?

Use Python to compute quantiles and probabilities from relevant probability distribution(s).

b) Compute a 95% confidence interval for the difference in mean film thickness between Supplier 1 and Supplier 4 (as a single pre-planned comparison).

Use Python to compute quantiles and probabilities from relevant probability distribution(s).

8.4 Brass alloys

|||| Exercise 8.4 Brass alloys

When brass is used in a production, the modulus of elasticity, E, of the material is often important for the functionality. The modulus of elasticity for 6 different brass alloys is measured. 5 samples from each alloy are tested. The results are shown in the table below where the measured modulus of elasticity is given in GPa:

Brass alloys					
M1	M2	M3	M4	M5	M6
82.5	82.7	92.2	96.5	88.9	75.6
83.7	81.9	106.8	93.8	89.2	78.1
80.9	78.9	104.6	92.1	94.2	92.2
95.2	83.6	94.5	87.4	91.4	87.3
80.8	78.6	100.7	89.6	90.1	83.8

A one-way ANOVA gives the following output (some values replaced by A, B, C):

```
           Df  Sum Sq  Mean Sq  F value     Pr(>F)
alloy       A  1192.51  238.501  9.9446  3.007e-05
Residuals   B       C   23.983      NaN        NaN
```

a) What are the values of A, B, and C?

b) Which of the following gives the complete set of assumptions for a one-way ANOVA?

1) Observations are independent, the data in each group are normally distributed, and the group variances are equal.
2) Observations are independent and the data in each group are normally distributed, and the group means are equal.
3) The sample sizes are independent and normally distributed.
4) The sample size in each group is neither too large nor too small.
5) Observations are independent, the data in each group are normally distributed, and the groups have approximately the same interquartile range (IQR).

c) Compute a 95% confidence interval for the single pre-planned difference between brass alloys M1 and M2.

Use Python to compute quantiles and probabilities from relevant probability distribution(s).

8.5 Plastic tubes

Exercise 8.5 Plastic tubes

Plastic tubes are produced using two different types of granules (base materials), from four different suppliers. The tensile strength of the tubes is measured in MPa and recorded for each combination of supplier and granule type. The measurements from this trial are shown below:

	Granule	
	g1	g2
Supplier a	34.2	33.1
Supplier b	34.8	31.2
Supplier c	31.3	30.2
Supplier d	31.9	31.6

The following Python code fits a two-way ANOVA model with supplier and granule type as categorical factors, then prints the ANOVA table. (Optionally, try to tun the code in your Python notebook to reproduce the results)

```
D = pd.DataFrame({
    "strength": [34.2, 34.8, 31.3, 31.9,
                 33.1, 31.2, 30.2, 31.6],
    "supplier": pd.Categorical(["a", "b", "c", "d",
                                "a", "b", "c", "d"]),
    "granule": pd.Categorical([1, 1, 1, 1, 2, 2, 2, 2])
})

fit = smf.ols("strength ~ supplier + granule", data=D).fit()
print(sm.stats.anova_lm(fit))
```

The output of the two-way ANOVA is:

```
            df     sum_sq   mean_sq          F    PR(>F)
supplier   3.0  10.03375  3.344583  3.253749  0.179225
granule    1.0   4.65125  4.651250  4.524929  0.123339
Residual   3.0   3.08375  1.027917       NaN       NaN
```

a) Which distribution was used to find the p-value 0.1792?

b) Which conclusion is best supported by the two-way ANOVA results, at a significance level of $\alpha = 0.05$?

1) A significant difference has been found between the tensile-strength variances.
2) A significant difference in mean tensile strength has been found between granule types, but not between suppliers.
3) No significant difference in mean tensile strength has been found between suppliers or between granule types.
4) Significant differences in mean tensile strength have been found for both suppliers and granule types.
5) A significant difference in mean tensile strength has been found between suppliers, but not between granule types.

Exam-style questions

c) Supplier c has a mean tensile strength of 30.75 MPa, which is evidently lower than the mean tensile strength of 33.00 MPa for supplier b. However, the supplier effect is not statistically significant ($p = 0.1792$). Given that there is only one observation per combination of supplier-granule, which statement about this result is correct?

1) The non-significant result shows that the observed difference between suppliers b and c is due to sampling variation rather than a real supplier effect.
2) Differences between granule types remain in the residual variation, making differences between suppliers harder to distinguish from random error.
3) Without repeated measurements for each combination, the error variance cannot be estimated, so the supplier test has no residual degrees of freedom.
4) The error variance is estimated with only three residual degrees of freedom, which can limit the power to detect supplier differences.
5) The supplier test requires significant differences between every pair of suppliers, making an overall significant supplier effect harder to establish.

8.6 Joining methods

|||| Exercise 8.6 Joining methods

A series of experiments is carried out to investigate how joining method and material affect the strength of a joint. Three joining methods (A, B, C) are tested with four materials (1, 2, 3, 4). One strength measurement is recorded for each combination of joining method and material. The measurements are shown below, together with the row and column averages:

	Material				Row
Joining method	1	2	3	4	average
A	242	214	254	248	239.50
B	248	214	248	247	239.25
C	236	211	245	243	233.75
Column average	242	213	249	246	

A two-way ANOVA model is fitted to the data. The Python code used for this analysis produces the output shown below. Six numerical values have been replaced by the symbols A-F:

```
                df   sum_sq   mean_sq          F    PR(>F)
Joiningmethod    A    84.5         B          C  0.050408
Material         D       E   825.00          F  0.000016
Residual       6.0    49.5      8.25        NaN       NaN
```

a) Calculate the missing values A, B and C in the ANOVA table.

b) What are the conclusions regarding the importance of the two factors ($\alpha = 5\%$)?

c) Perform a post hoc analysis for materials, and for joining methods. Make pairwise comparisons of mean strength using confidence intervals, hypothesis tests, or both. Apply the Bonferroni correction with an overall significance level of $\alpha = 0.05$ for each factor.

d) Perform a residual analysis to check the model assumptions:

1. Normality
2. Variance homogeneity

Exam-style question

e) The two-way ANOVA gives a p-value of approximately 0.05041 for joining method. Which statement best describes this result at a significance level of $\alpha = 0.05$?

1) The result establishes that the joining methods have equal population mean strengths, since the null hypothesis is not rejected.
2) There is weak evidence of differences in mean strength between joining methods, but the result is not statistically significant at the 5% level.
3) There is approximately a 5.041% probability that the population mean strength is the same for all three joining methods.
4) A significant difference in mean strength can be concluded, since the p-value is sufficiently close to the 5% significance level.
5) The p-value must first be compared with $0.05/3$, since testing the joining-method factor involves comparing three joining methods.

8.7 Remoulade

|||| Exercise 8.7 Remoulade

A supermarket wants to find the best recipe for homemade "remoulade" sauce. A taste experiment is conducted with four dressing types (A, B, C, D) and three pickle types (I, II, III). One taste score is recorded for each combination of pickle type and dressing type, using a continuous scale from 0 to 5. The scores are shown below, together with the row and column averages:

	Dressing type				Row
Pickle type	A	B	C	D	average
I	4.0	3.0	3.8	2.4	3.30
II	4.3	3.1	3.3	1.9	3.15
III	3.9	2.3	3.0	2.4	2.90
Column average	4.07	2.80	3.37	2.23	

A two-way ANOVA model is fitted to the data. In the Python output below, six numerical values have been replaced by the symbols A-F:

```
            df     sum_sq   mean_sq          F    PR(>F)
Pickles      A   0.326667  0.163333          E  0.287133
Dressing     B   5.536667  1.845556          F  0.002273
Residual     C          D  0.105556        NaN       NaN
```

a) Calculate the missing values A, B and C in the ANOVA table.

b) Calculate the missing values D, E and F in the ANOVA table.

c) At a significance level of $\alpha = 0.05$, what can be concluded about the effects of dressing type and pickle type on mean taste score?

Exam-style question

d) A similar taste experiment compares five pickle types and six dressing types, with one observation for each combination. An additive two-way ANOVA model is fitted to the data.
What are the degrees of freedom for pickle type, dressing type and residuals, respectively?

1) 4, 5, 19
2) 5, 6, 18
3) 4, 5, 9
4) 4, 5, 29
5) 4, 5, 20

8.8 Transport times

||| Exercise 8.8 Transport times

Delivery times are compared for three transport firms (A, B, C) and three item sizes (small, intermediate and large). One delivery time is recorded in days for each combination of firm and item size. The measurements are shown below, together with the row and column averages:

	Item size			Row
	Small	Intermediate	Large	average
Firm A	1.4	2.5	2.1	2.00
Firm B	0.8	1.8	1.9	1.50
Firm C	1.6	2.0	2.4	2.00
Column average	1.27	2.10	2.13	

A two-way ANOVA model is fitted to the data. In the Python output below, four numerical values have been replaced by the symbols A-D:

```
            df     sum_sq    mean_sq           F    PR(>F)
Firm       2.0          A          B    4.285714   0.10124
Itemsize   2.0   1.446667          C           D   0.01929
Residual   4.0   0.233333   0.058333         NaN       NaN
```

a) Calculate the missing values A, B, C and D in the ANOVA table.

b) At a significance level of $\alpha = 0.05$, what can be concluded about the effects of transport firm and item size on mean delivery time?

Exam-style question

c) A similar delivery-time study compares three transport firms and four item-size categories, with one observation for each combination. An additive two-way ANOVA gives $MSE = 0.08$ and an F-statistic of 5.0 for firm.

What is the mean square for firm, and how many residual degrees of freedom does the model have?

1) $MS(\text{Firm}) = 0.016$, residual $df = 6$
2) $MS(\text{Firm}) = 0.4$, residual $df = 12$
3) $MS(\text{Firm}) = 0.4$, residual $df = 6$
4) $MS(\text{Firm}) = 0.016$, residual $df = 5$
5) $MS(\text{Firm}) = 0.4$, residual $df = 5$