Chapter 4

Statistics by Simulation (solutions to exercises)

Contents

Import Python packages

```
# Import all needed python packages
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import scipy.stats as stats
import statsmodels.formula.api as smf
import statsmodels.api as sm
```

4.1 Reliability: System lifetime (simulation as a computation tool)

|||| Exercise 4.1 Reliability: System lifetime (simulation as a computation tool)

A system consists of three components A, B and C serially connected, such that A is positioned before B, which is again positioned before C. The system will be functioning only so long as A, B and C are all functioning. The lifetime in months of the three components are assumed to follow exponential distributions with means: 2 months, 3 months and 5 months, respectively (hence there are three random variables, X_A, X_B and X_C with exponential distributions with $\lambda_A = 1/2$, $\lambda_B = 1/3$ and $\lambda_C = 1/5$ resp.). A little Python-help: You will probably need (or at least it would help) to put three variables together to make e.g. a $k \times 3$-matrix – this can be done by the `cbind` function:

```
x = np.column_stack((xA,xB,xC))
```

And just as an example, in Python we can easily compute e.g. the mean of the three values for each of all the k rows of this matrix by using the "axis" argument in the `np.mean` function. This argument specifies the axis along which the means are computed, so for axis=1, we take the mean for the three columns in the x. Many functions in Python have this argument, so it is a good idea to get familiar with it. Example for mean:

```
simmeans = np.mean(x, axis=1)
```

a) Generate, by simulation, a large number (at least 1000 – go for 10000 or 100000 if your computer is up for it) of system lifetimes (hint: consider how the random variable Y = System lifetime is a function of the three X-variables: is it the sum, the mean, the median, the minimum, the maximum, the range or something even different?).

|||| Solution

Note that the lifetime can be seen as the minimal value of the three random component lifetimes:

$$\text{"Lifetime"} = \min(X_A, X_B, X_C).$$

First, note that the generated solution below has been generated with this seed in order to get the same result each time. Note, that when a simulation analysis is carried out, this number should only be set once and set randomly (potentially it is possible to find a seed (see Remark 2.12) that gives a rare simulation result and thus showing a "wrong" result, however if k is high enough this is very hard). The solution below has been generated with the following seed

```
## You might want to set the seed to achieve a particular result
np.random.seed(82719)
```

The following Python-code generates 10.000 simulated system lifetimes:

```
# Number of simulations
k = 10000
# Generating k component A lifetimes
xA = np.random.exponential(scale=2, size=k)
# Checking the mean of these
print(np.mean(xA))
```

```
2.0035078916253752
```

```
# Generating k component B lifetimes
xB = np.random.exponential(scale=3, size=k)
# Checking the mean of these
print(np.mean(xB))
```

```
3.0264519724715777
```

```
# Generating k component C lifetimes
xC = np.random.exponential(scale=5, size=k)
# Checking the mean of these
print(np.mean(xC))
```

```
5.047803906911606
```

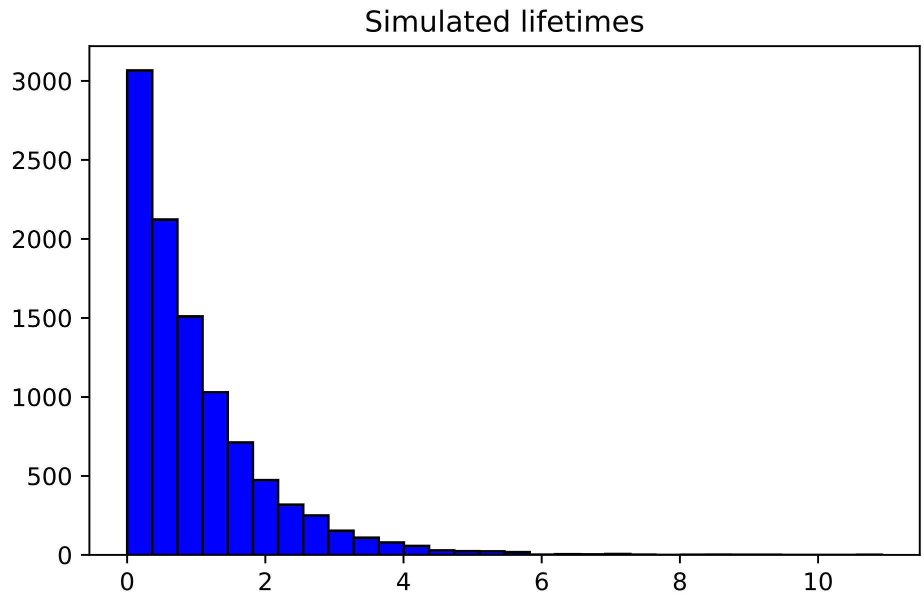
```
# Putting these three sets of k lifetimes together
# into a single k-by-3 matrix
x = np.column_stack((xA, xB, xC))

# Finding the minimum value of the three components in each of the k
situations
lifetimes = np.min(x, axis=1)
```

|||| Solution

Let us have a look at these simulated lifetimes:

```
## Histogram of the simulated lifetimes
plt.hist(lifetimes, bins=30, color='blue', edgecolor='black')
plt.title('Simulated lifetimes')
plt.show()
```



b) Estimate the mean system lifetime.

|||| Solution

```
## The estimated mean lifetime
print(np.mean(lifetimes))

0.9880015386365172
```

c) Estimate the standard deviation of system lifetimes.

|||| Solution

```
## The estimated std. dev. of the lifetime
print(np.std(lifetimes,ddof=1))

0.9959225338611547
```

d) Estimate the probability that the system fails within 1 month.

|||| Solution

We need to count how often the lifetimes are smaller than or equal to 1 month – this can in Python be achieved by use of a logical operator:

```
## The fraction of times the simulated lifetime was below or equal 1
print(np.mean(lifetimes <= 1))

0.6391

# or
print(np.sum(lifetimes <= 1)/k)

0.6391
```

In Python `FALSE` is a 0 and a `TRUE` is a 1 - this is why we can simply apply the `mean` function directly on the vector of TRUES and FALSES like this.

e) Estimate the median system lifetime

|||| Solution

```
## The estimated median lifetime
print(np.median(lifetimes))
```

```
0.6878542614451192
```

f) Estimate the 10th percentile of system lifetimes

|||| Solution

```
## The estimated 10% quantile
print(np.quantile(lifetimes, 0.1))
```

```
0.10367217182567331
```

g) What seems to be the distribution of system lifetimes? (histogram etc)

|||| Solution

We already made the histogram above. It appears that the minimum of the three exponential variables also has a distribution that looks like an exponential. In fact, there is a theoretical result (beoynd the syllabus of this course) that states that the distribution of the minimum of these three exponential distributions is again an exponential distribution but now with

$$\lambda_{min} = \lambda_A + \lambda_B + \lambda_C = 1/2 + 1/3 + 1/5 = 31/30.$$

Note how this matches nicely with the found mean above!

4.2 Basic bootstrap CI

Exercise 4.2 Basic bootstrap CI

(Can be handled without using R) The following measurements were given for the cylindrical compressive strength (in MPa) for 11 prestressed concrete beams:

$$38.43, 38.43, 38.39, 38.83, 38.45, 38.35, 38.43, 38.31, 38.32, 38.48, 38.50.$$

1000 bootstrap samples (each sample hence consisting of 11 measurements) were generated from these data, and the 1000 bootstrap means were arranged on order. Refer to the smallest as $\bar{x}^*_{(1)}$, the second smallest as $\bar{x}^*_{(2)}$ and so on, with the largest being $\bar{x}^*_{(1000)}$. Assume that

$$\begin{aligned}
\bar{x}^*_{(25)} &= 38.3818,\\
\bar{x}^*_{(26)} &= 38.3818,\\
\bar{x}^*_{(50)} &= 38.3909,\\
\bar{x}^*_{(51)} &= 38.3918,\\
\bar{x}^*_{(950)} &= 38.5218,\\
\bar{x}^*_{(951)} &= 38.5236,\\
\bar{x}^*_{(975)} &= 38.5382,\\
\bar{x}^*_{(976)} &= 38.5391.
\end{aligned}$$

a) Compute a 95% bootstrap confidence interval for the mean compressive strength.

Solution

Looking at Method box 4.10, we see that we need to find the 2.5%, and 97.5% quantiles of the 1000 bootstrap samples. According to the rule for finding the 2.5% quantile this should be the average of the 25th andn the 26th observation:

$$q_{0.025} = \frac{\bar{x}^*_{(25)} + \bar{x}^*_{(26)}}{2} = 38.3818,$$

and similarly

$$q_{0.975} = \frac{\bar{x}^*_{(975)} + \bar{x}^*_{(976)}}{2} = \frac{38.5382 + 38.5391}{2} = 38.5387,$$

and hence the 95% bootstrap confidence band is:

$$[38.3818; 38.5387].$$

b) Compute a 90% bootstrap confidence interval for the mean compressive strength.

|||| Solution

As above we get:

$$q_{0.05} = \frac{\bar{x}^*_{(50)} + \bar{x}^*_{(51)}}{2} = \frac{38.3909 + 38.3919}{2} = 38.3914,$$

and similarly:

$$q_{0.95} = \frac{\bar{x}^*_{(950)} + \bar{x}^*_{(951)}}{2} = \frac{38.5218 + 38.5236}{2} = 38.5227,$$

and hence the 90% bootstrap confidence band is:

$$[38.3914; 38.5227].$$

4.3 Various bootstrap CIs

Exercise 4.3 Various bootstrap CIs

Consider the data from the exercise above. These data are entered into Python as:

```
x = np.array([38.43, 38.43, 38.39, 38.83, 38.45, 38.35,
              38.43, 38.31, 38.32, 38.48, 38.50])
```

Now generate $k = 1000$ bootstrap samples and compute the 1000 means (go higher if your computer is fine with it)

a) What are the 2.5%, and 97.5% quantiles (so what is the 95% confidence interval for μ without assuming any distribution)?

Solution

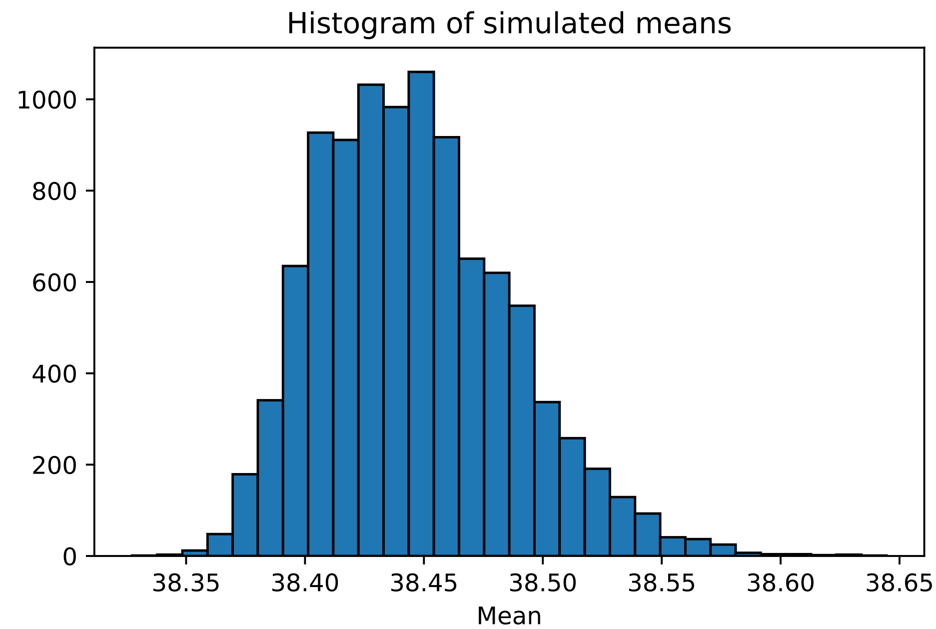
The solution below has been generated with the following seed (see Remark 2.12)

```
## You might want to set the seed to achieve a particular result
np.random.seed(6287)
```

```
x = np.array([38.43, 38.43, 38.39, 38.83, 38.45, 38.35,
              38.43, 38.31, 38.32, 38.48, 38.50])
k = 10000
n = len(x)
simsamples = np.random.choice(x, (n, k), replace=True)
simmeans = np.mean(simsamples, axis=0)
print(np.quantile(simmeans, [0.025, 0.975]))

[38.381 38.536]
```

```
plt.hist(simmeans, bins=30, edgecolor = 'black')
plt.title('Histogram of simulated means')
plt.xlabel('Mean')
plt.show()
```



b) Find the 95% confidence interval for μ by the parametric bootstrap assuming the normal distribution for the observations. Compare with the classical analytic approach based on the t-distribution from Chapter 2.

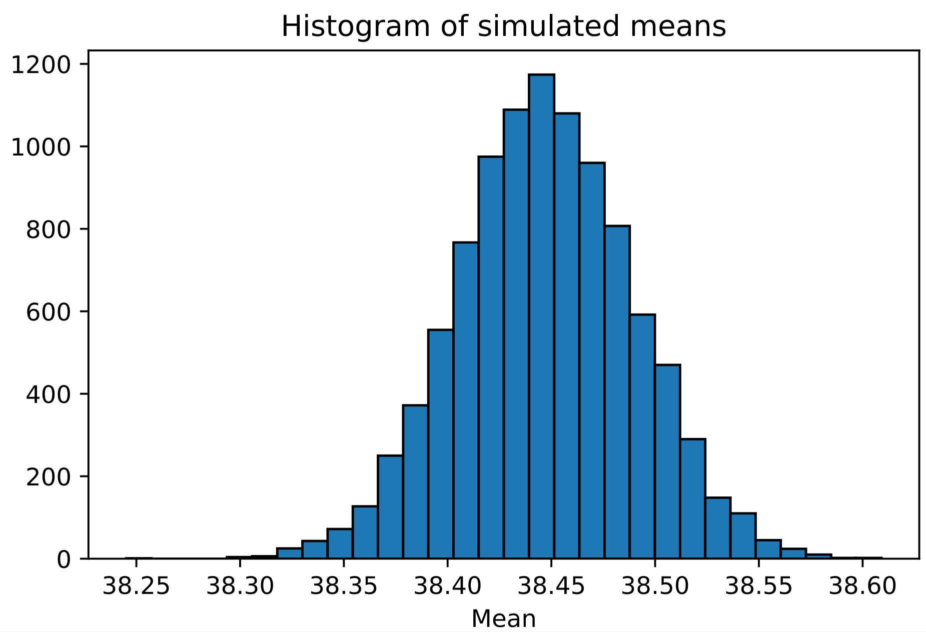
|||| Solution

First we do the parametric bootstrap:

```
k = 10000
n = len(x)
simsamples = np.random.normal(np.mean(x), np.std(x,ddof=1), (n, k))
simmeans = np.mean(simsamples, axis=0)
print(np.quantile(simmeans, [0.025, 0.975]))
```

```
[38.363 38.530]
```

```
# Histogram
plt.hist(simmeans, bins=30, edgecolor = 'black')
plt.title('Histogram of simulated means')
plt.xlabel('Mean')
plt.show()
```



And the classic t-based approach (without simulation):

```
t_stat,p_val = stats.ttest_1samp(x, 38.5)
print(t_stat)

-1.239610578766898


print(p_val)

0.24342150717016434


# interval directly
(CI_low,CI_high) = stats.t.interval(0.95, len(x)-1, loc=np.mean(x),
scale=stats.sem(x))
print(CI_low,CI_high)

38.35249805615088 38.54204739839457
```

c) Find the 95% confidence interval for μ by the parametric bootstrap assuming the log-normal distribution for the observations. (Help: To use the `np.random.lognormal` function to simulate the log-normal distribution, we face the challenge that we need to specify the mean and standard deviation on the log-scale and not on the raw scale, so compute mean and standard deviation for log-transformed data for this Python-function)

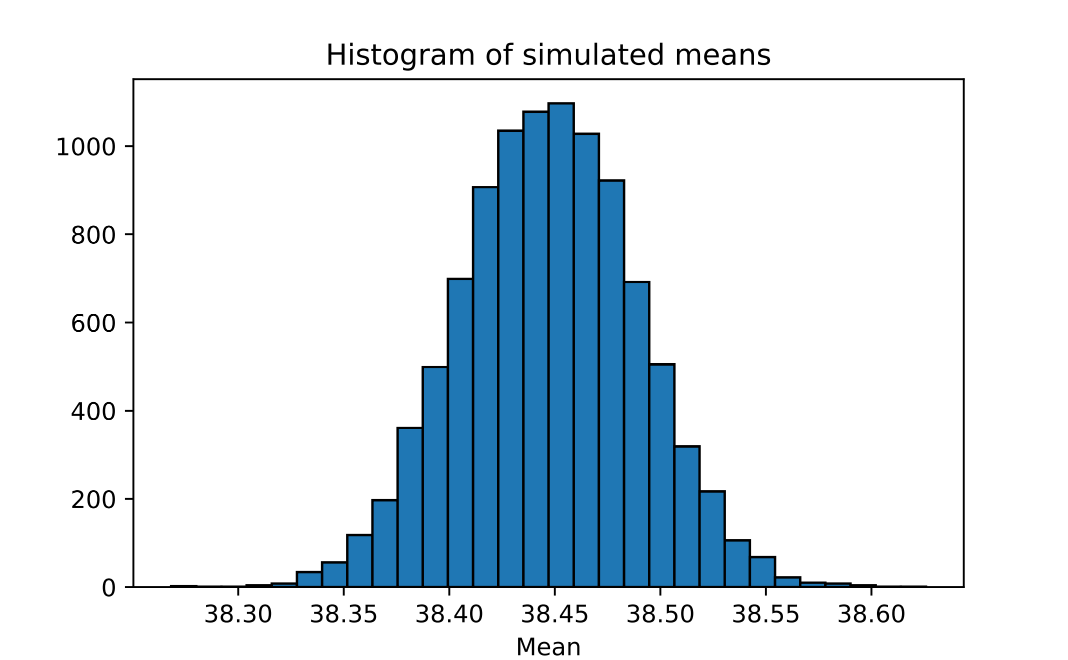
|||| Solution

We do the parametric bootstrap using the log-normal distribution.

```
k = 10000
n = len(x)
simsamples = np.random.lognormal(np.mean(np.log(x)),
np.std(np.log(x),ddof=1), (n, k))
simmeans = np.mean(simsamples, axis=0)
print(np.quantile(simmeans, [0.025, 0.975]))
```

```
[38.365 38.529]
```

```
# Histogram
plt.hist(simmeans, bins=30, edgecolor = 'black')
plt.title('Histogram of simulated means')
plt.xlabel('Mean')
plt.show()
```



d) Find the 95% confidence interval for the lower quartile Q_1 by the parametric bootstrap assuming the normal distribution for the observations.

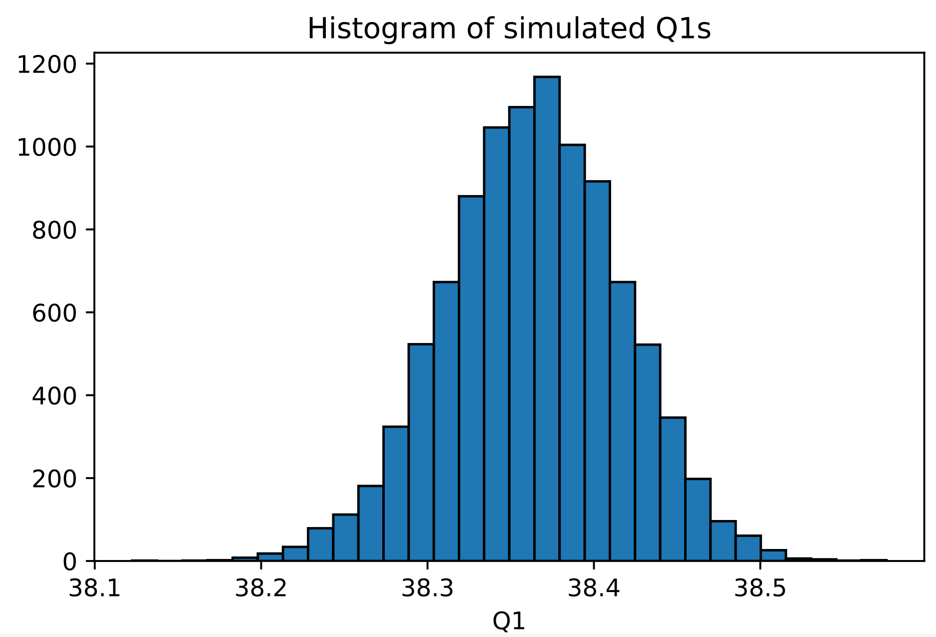
Solution

We do the parametric bootstrap of lower quartile Q_1 using the normal distribution by first making a $Q1$-function in Python, and then the usual stuff:

```
k = 10000
n = len(x)
simsamples = np.random.normal(np.mean(x),np.std(x,ddof=1), (n, k))
simQ1s = np.quantile(simsamples, 0.25, axis=0)
print(np.quantile(simQ1s, [0.025, 0.975]))
```

```
[38.258 38.466]
```

```
# Histogram
plt.hist(simQ1s, bins=30, edgecolor = 'black')
plt.title('Histogram of simulated Q1s')
plt.xlabel('Q1')
plt.show()
```



e) Find the 95% confidence interval for the lower quartile Q_1 by the non-parametric bootstrap (so without any distributional assumptions)

Solution

We simply substitute the sampling line with the non-parametric version:

```
k = 10000
n = len(x)
simsamples = np.random.choice(x, (n, k), replace=True)
simQ1s = np.quantile(simsamples, 0.25, axis=0)
print(np.quantile(simQ1s, [0.025, 0.975]))
```

```
[38.315 38.430]
```

4.4 Two-sample TV data

Exercise 4.4 Two-sample TV data

A TV producer had 20 consumers evaluate the quality of two different TV flat screens - 10 consumers for each screen. A scale from 1 (worst) up to 5 (best) were used and the following results were obtained:

TV screen 1	TV screen 2
1	3
2	4
1	2
3	4
2	2
1	3
2	2
3	4
1	3
1	2

a) Compare the two means without assuming any distribution for the two samples (non-parametric bootstrap confidence interval and relevant hypothesis test interpretation).

|||| Solution

The solution below has been generated with the following seed (see Remark 2.12)

```
## You might want to set the seed to achieve a particular result
np.random.seed(98273)
```

```
x1 = np.array([1, 2, 1, 3, 2, 1, 2, 3, 1, 1])
x2 = np.array([3, 4, 2, 4, 2, 3, 2, 4, 3, 2])
## Number of simulated (bootstrapped) samples
k = 10000
n = len(x1) # same as len(x2)
## Simulated samples of TV1 and TV2 groups
simx1samples = np.random.choice(x1, (n, k), replace=True)
simx2samples = np.random.choice(x2, (n, k), replace=True)
simmeandifs = np.mean(simx1samples, axis=0) - np.mean(simx2samples,
axis=0)
# Confidence interval
ci = np.quantile(simmeandifs, [0.025, 0.975])
print(ci)

[-1.900 -0.500]
```

We reject the null hypothesis of $\mu_1 = \mu_2$, since zero is not included in the CI of the differences.

b) Compare the two means assuming normal distributions for the two samples - without using simulations (or rather: assuming/hoping that the sample sizes are large enough to make the results approximately valid).

|||| Solution

```
t_stat, p_val = stats.ttest_ind(x1, x2)
print(t_stat)

-3.157408869505305


print(p_val)

0.005449057981469947
```

We reject the null hypothesis of $\mu_1 = \mu_2$.

c) Compare the two means assuming normal distributions for the two samples - simulation based (parametric bootstrap confidence interval and relevant hypothesis test interpretation – in spite of the obviously wrong assumption).

|||| Solution

```
simx1samples = np.random.normal(np.mean(x1), np.std(x1,ddof=1), (n, k))
simx2samples = np.random.normal(np.mean(x2), np.std(x2,ddof=1), (n, k))
simmeandifs = np.mean(simx1samples, axis=0) - np.mean(simx2samples,
axis=0)
# Confidence interval
print(np.quantile(simmeandifs, [0.025, 0.975]))

[-1.948 -0.442]
```

We reject the null hypothesis of $\mu_1 = \mu_2$.

4.5 Non-linear error propagation

Exercise 4.5 Non-linear error propagation

The pressure P, and the volume V of one mole of an ideal gas are related by the equation $PV = 8.31T$, when P is measured in kilopascals, T is measured in kelvins, and V is measured in liters.

a) Assume that P is measured to be 240.48 kPa and V to be 9.987 L with known measurement errors (given as standard deviations): 0.03 kPa and 0.002 L. Estimate T and find the uncertainty in the estimate.

Solution

This is a almost direct copy of the rectangle example ($A = XY$) (Example 4.5), since $T = PV/8.31$, so since: To use the approximate error propagation rule, we must differentiate the function $f(x, y) = xy/8.31$ with respect to both x and y:

$$\frac{\partial f}{\partial x} = y/8.31 \quad \frac{\partial f}{\partial y} = x/8.31.$$

We get the result: $\hat{T} = 240.48 \cdot 9.987/8.31 = 289.0101$, and the uncertainty is:

$$\sigma_{\hat{T}} = \sqrt{9.987^2 \times 0.03^2 + 240.48^2 \times 0.002^2}/8.31 = 0.0682.$$

b) Assume that P is measured to be 240.48kPa and T to be 289.12K with known measurement errors (given as standard deviations): 0.03kPa and 0.02K. Estimate V and find the uncertainty in the estimate.

|||| Solution

$$V = f(P, T) = 8.31T/P.$$

So:

$$\frac{\partial f}{\partial T} = 8.31/P \quad \frac{\partial f}{\partial P} = -8.31\frac{T}{P^2},$$

and hence:

$$\hat{V} = 8.31 \cdot 289.12/240.48 = 9.9908.$$

and

$$\sigma_{\hat{V}} = 8.31\sqrt{1/240.48^2 \times 0.02^2 + 289.12^2/240.48^4 \times 0.03^2} = 0.00143.$$

c) Assume that V is measured to be 9.987 L and T to be 289.12 K with known measurement errors (given as standard deviations): 0.002 L and 0.02 K. Estimate P and find the uncertainty in the estimate.

|||| Solution

Since

$$P = f(V, T) = 8.31T/V,$$

we can simply change the roles of P and V in the above and find similarly

$$\frac{\partial f}{\partial T} = 8.31/V \quad \frac{\partial f}{\partial V} = -8.31\frac{T}{V^2},$$

and hence

$$\hat{P} = 8.31 \cdot 289.12/9.987 = 240.5715,$$

and

$$\sigma_{\hat{P}} = 8.31\sqrt{1/9.987^2 \times 0.02^2 + 289.12^2/9.987^4 \times 0.002^2} = 0.0510.$$

d) Try to answer one or more of these questions by simulation (assume that the errors are normally distributed).

|||| Solution

Let's look at 3. The following Python-code will do the job:

The solution below has been generated with the following seed (see Remark 2.12)

```
## You might want to set the seed to achieve a particular result
np.random.seed(28973)
```

```
k = 10000
Vs = np.random.normal(loc=9.987, scale=0.002,size=k)
Ts = np.random.normal(loc=289.12,scale=0.02, size=k)
Ps = 8.31*Ts/Vs
print(np.std(Ps, ddof=1))
```

```
0.05119900268933971
```

Rerunning this a few times will show that 0.051 is the proper result. This additional re-running gives a feeling of the error in the simulation - rather small here. Alternatively increase k.

Similarly 2. can be handled as:

```
k = 10000
Ps = np.random.normal(240.28, 0.03, k)
Ts = np.random.normal(loc=289.12,scale=0.02, size=k)
Vs = 8.31*Ts/Ps
print(np.std(Vs, ddof=1))
```

```
0.0014176700426727481
```

Providing again basically the same answer as above: 0.0014.